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PLASTIC CONSTANTS OF FRACTURE

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Based on the theory of an ideal rigid-plastic body, an approach is formulated for determining fracture constants on the basis of standard mechanical tests on uniaxial extension of plane and cylindrical samples. Instead of the experimentally determined characteristics of fracture of materials (dimensionless elongation and constriction of the sample during its fracture), two invariant tensor characteristics of the degree of sample deformation are introduced, which correspond to the moment of origination of a macrocrack and critical strain at the crack tip determining the process of crack propagation.

Key words: *deformation, plasticity, fracture.*

Introduction. Important problems of the theory of ideal rigid-plastic bodies are the uncertainty of the position and type of the plastic region and the nonuniqueness of the field of displacement velocities responsible for changes in the body geometry [1]. For practical application of theoretical solutions, one needs criteria for choosing a preferable instantaneous field of displacement velocities and criteria determining the time evolution of the velocity field (changes in the plastic region). Such criteria have not been formulated previously in the theory of an ideal rigid-plastic body.

The criteria for choosing a preferable plastic flow can be related to the specific features of the formulation of the extreme principles of nonequilibrium thermodynamics on discontinuities of the field of displacement velocities.

The governing principle in constructing the plasticity theory is the principle of the maximum dissipation rate of mechanical work (Mises' maximum principle, one of the extreme principles), which is related to the principle proposed by Onsager [1]. Mises' principle is usually formulated in terms of the powers of dissipation of mechanical work. From Mises' principle, there follows the associated law of the flow, the theorem of uniqueness for the stress field, and the possibility of existence of discontinuities in the field of displacement velocities. On the discontinuities of the field of displacement velocities, however, Mises' principle is not satisfied, i.e., the specific power of energy dissipation cannot be determined, as the components of the strain-rate tensor turn to infinity or are not determined. Therefore, the extreme principles of thermodynamics imposing constraints on dissipative properties of materials should be formulated with allowance for the specific features of the fields of displacement velocities. In the present paper, we propose to relate these principles to strains in material particles and specific dissipation of energy, i.e., actually, to use empirical generalization of the extreme principles of nonequilibrium thermodynamics formulated in [2] as the principle of the minimum energy dissipation.

Determining Strain Fields. As a measure of strain, we use the Almansi tensor of finite strains E, which is determined via the distortion tensor A as

$$E_{ij} = (\delta_{ij} - A_{ki}A_{kj})/2, \qquad A_{ij} = x^0_{j,i}, \qquad i,j = 1,2,3. \tag{1}$$

Here δ_{ij} is the Kronecker symbol and x^0_i and x_i are the Lagrangian and Eulerian coordinates of the particle, respectively. The changes in these tensors along the particle trajectory are described by the equations (see [3])

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$$\frac{d}{dt}A + W^*A = 0, \qquad \frac{dE}{dt} + EW + W^*E = \varepsilon, \tag{2}$$

where $d/dt = \partial/\partial t + V_k\,\partial/\partial x_k$, $W_{ik} = V_{i,k}$, and $W^*_{ki} = V_{k,i}$.

The choice of the Almansi strain tensor as a measure of strain is not the only possible one, because equations of the type (2) can be also obtained for other strain tensors (see [3]).

The field of displacement velocities can have some singularities (discontinuity surfaces V_i, center of the fan of characteristics). Therefore, strain accumulation occurs under two conditions: 1) in a continuous field of velocities V_i in accordance with Eqs. (2); 2) when the material particle crosses the singularities of the field V_i at which the components ε_{ij} can turn to infinity.

The change in strains caused by the material particle crossing the singularities of the velocity field V_i was considered in [4–7]. Based on the theory of discontinuities proposed by Hadamard and Thomas [8], the changes in distortion components on the surfaces of discontinuities of the field of displacement velocities are determined by the expressions

$$[x^0_{i,j}] = \frac{[V_\tau]}{G + V_\nu}\,\tau_i\nu_j, \qquad x^{0+}_{i,j} = \delta_{ij} + \frac{[V_\tau]}{G + V_\nu}\,\tau_i\nu_j \tag{3}$$

under the assumption that the material has not been deformed before the discontinuity surface is crossed ($x^0_{i,j} = \delta_{ij}$). Here $[V_\tau]$ and V_ν are the discontinuity of the tangential component and the normal component of the displacement velocity on the discontinuity surface, G is the normal velocity of propagation of the discontinuity surface, τ_i is the unit vector of the tangential line to the discontinuity surface, which coincides with the vector of the discontinuity of displacement velocities, and ν_i is the unit vector normal to the discontinuity surface.

Let us clear up the physical essence of the quantity $W = [V_\tau]/(G + V_\nu)$. The expression $[V_\tau]k\,dS\,dt$ describes the elementary work of shear forces on the discontinuity surface during the time dt, where dS is the element of the discontinuity-surface area and k is the tangential component of stresses on the discontinuity surface (or the yield point for an ideal rigid-plastic body). The expression $(G + V_\nu)\,dS\,dt$ describes the volume of the material passing through the surface element dS during the time dt. Then, the absolute value of the quantity

$$H = \frac{[V_\tau]}{G + V_\nu}\,k \tag{4}$$

has the physical meaning of volume density dissipation of energy acquired by a material particle crossing the surface of discontinuity of displacement velocities, $W = H/k$.

It follows from Eqs. (1), (3), and (4) that the principal invariants of the tensor E_{ij} are calculated (for $x^0_{i,j} = \delta_{ij}$) via the quantity W by the formulas

$$\begin{gathered}(I_1)_E = \frac{1}{2}(E_{11} + E_{22}) = -\frac{W^2}{4}, \quad (I_2)_E = (E_{11} - E_{22})^2 + 4E_{12}^2 = \frac{W^2}{4}(W^2 + 4), \quad (I_3)_E = 0,\\ E_1 = \frac{W^2}{4}\Big(\sqrt{1 + \frac{4}{W^2}} - 1\Big), \qquad E_2 = -\frac{W^2}{4}\Big(\sqrt{1 + \frac{4}{W^2}} + 1\Big), \qquad E_3 = 0.\end{gathered} \tag{5}$$

The relations obtained significantly depend on the motion of singularities of the velocity field with respect to material particles. All the parameters mentioned can be determined only by solving the problem with allowance for changes in the body geometry.

Under conditions of plane strain, owing to incompressibility of an ideal rigid-plastic body, only one invariant of the tensor E_{ij} is independent (e.g., E_1, which is the algebraically highest principal value) and can be used as a characteristic of the particle strain. The parameter E_1 is a monotonic function of W, and the quantity W can also characterize the magnitude of the strain of the particle crossing the line of discontinuity of displacement velocities.

If the material was deformed before crossing the discontinuity and the distortion-tensor components had the values $x^{0-}_{i,j}$, the distortion-tensor components behind the discontinuity have the values

$$x^{0+}_{i,j} = (\delta_{ik} + W\tau_i\nu_k)x^{0-}_{k,j}. \tag{6}$$

Criterion of Choosing a Preferable Plastic Flow. Let us formulate the following criterion.

1. The plastic flow is developed so that the maximum strain E_1 in the plastic region has the minimum value:

$$\inf_{d\Omega}\sup_{\Omega} E_1$$

(Ω are possible plastic regions for the full solutions of the problem and $d\Omega$ are possible changes in the plastic region due to the plastic flow determining the motion of its singularities at a given time).

Let us reformulate criterion 1 with allowance for Eq. (5).

2. The plastic flow is developed so that the maximum specific energy of dissipation in the plastic region has the minimum value:

$$\inf_{d\Omega} \sup_{\Omega} W.$$

Criteria 1 and 2 can be considered as one of the mathematical formulations of the local principle of the minimum energy dissipation.

Deformation of a Plane Sample. The following solutions are known for the problem of uniaxial extension of a plane sample: with a uniform strain field in the sample (Fig. 1a), with a discontinuous field of displacement velocities (Onat–Prager solution [9]; Fig. 1b), and possible formation of a double neck in a rather long sample (Fig. 1c). The hatched regions in Figs. 1, 3, and 4 correspond to regions of a deformed material.

In the case of a uniform field of displacement velocities, integration of Eqs. (4) yields the relations [10]

$$E_1 = \frac{2\delta + \delta^2}{2(\delta + 1)^2}, \qquad \frac{P}{4ka_0} = \frac{1}{\delta + 1}, \tag{7}$$

where $\delta = Vt/l_0$ is the dimensionless elongation of the strip,a_0 and l_0 are the initial width and length of the strip, respectively, P is the force necessary for sample deformation, and k is the yield point.

In the case of a discontinuous field of displacement velocities (Onat–Prager solution), we have

$$E_1 = W^2\Big(\sqrt{1 + 4/\bar{W}^2} - 1\Big), \qquad W = [V_\tau]/G + V_\nu,$$

$$[V_\tau] = \sqrt{2}\,V, \qquad V_\nu = \frac{V}{\sqrt{2}}, \qquad G = 0, \qquad \frac{P}{4ka_0} = 1 - \frac{V}{a_0}\,t.$$

Figure 2 shows the basic mechanical quantities characterizing the process of sample deformation [the first principal value of the Almansi strain tensor E_1 and the force $P/(4ka_0)$ necessary for sample deformation] as functions of the dimensionless elongation. These dependences show a significant difference in deformation processes: the strain in a continuous uniform field of displacement velocities increases with a finite velocity, uniformly over the entire sample; the strain in a discontinuous field changes in a jumplike manner by a finite value and is higher than the strain in a uniform field; in the latter case, only an insignificant part of the sample is subjected to strains. The work necessary for sample deformation in a uniform field of velocities is much higher than the work necessary for its deformation in a discontinuous field of velocities. Similar dependences are obtained in the case of axisymmetric deformation [7].

According to criteria 1 and 2, the process of sample deformation in a uniform field of displacement velocities is preferable, which is confirmed experimentally at the initial stage of deformation of plane and cylindrical samples up to the moment when a neck is formed [3].

Fracture of Ideal Rigid-Plastic Bodies. The existence of a preferable plastic flow allows us to formulate an approach to the description of fracture processes on the basis of a model of an ideal rigid-plastic body.

The strain field in the vicinity of the crack tip in the general case is nonuniform, and the strain tensor E_{ij} can be considered as a function of the coordinates φ and ρ of a polar coordinate system with the origin at the crack tip. Several lines of discontinuity of the strain tensor can approach the crack tip, i.e., the tensor components can be discontinuous in terms of the argument φ. Material fracture can be naturally associated with the tensor of strains accumulated during the entire history of material deformation. We can use the following simple model of fracture of an ideal rigid-plastic body.

$1'$. Fracture of the material begins when the maximum strain at the crack tip (E_1) reaches a critical value $\sup\limits_{\varphi} E_1 \geqslant E_*$. The velocity of motion of the crack tip is determined by the relation $E_1 = E_*$.

Condition $1'$ can be reformulated as follows.

$2'$. Fracture of the material begins when the accumulated specific dissipation of energy at the crack tip reaches a critical value $\sup\limits_{\varphi} W \geqslant W_*$. Then, the velocity of motion of the crack tip is determined by the relation $W = W_*$.

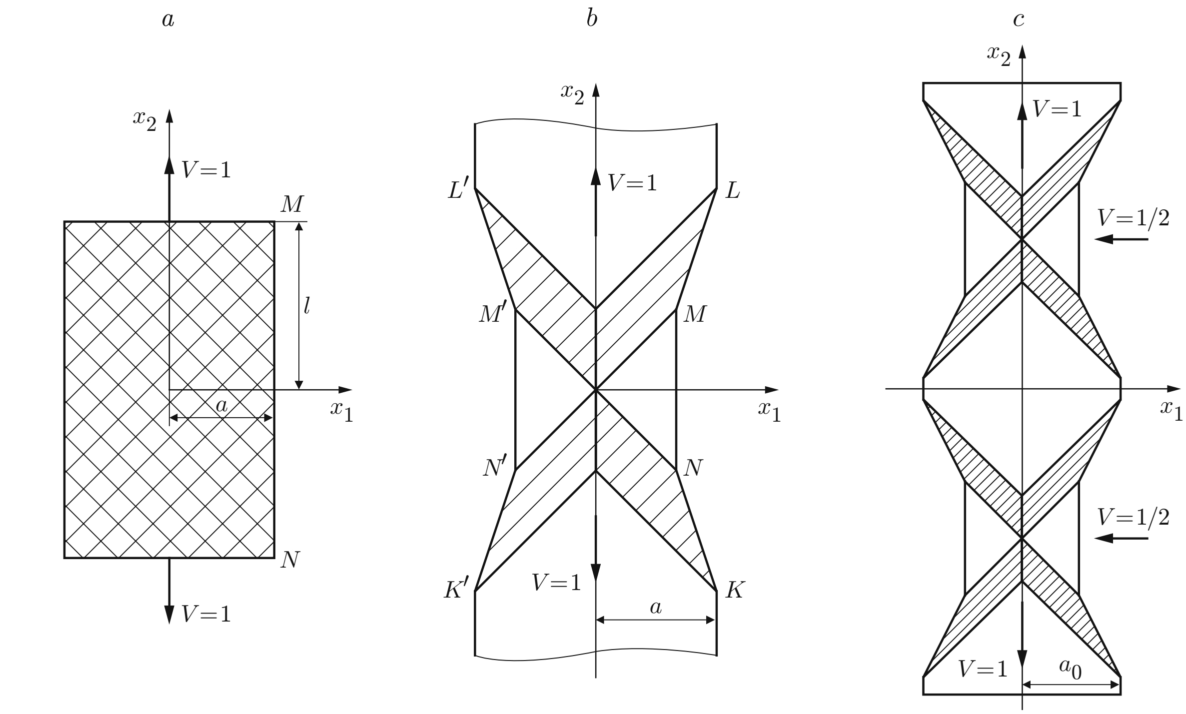

Fig. 1. Possible full solutions of the problem with uniaxial extension of a plane sample: plastic flow with a uniform continuous field of displacement velocities (a); plastic flow with a discontinuous field of displacement velocities: Onat–Prager solution [9] (b) and solution in the case of formation of a double neck predicted by Onat and Prager (c).

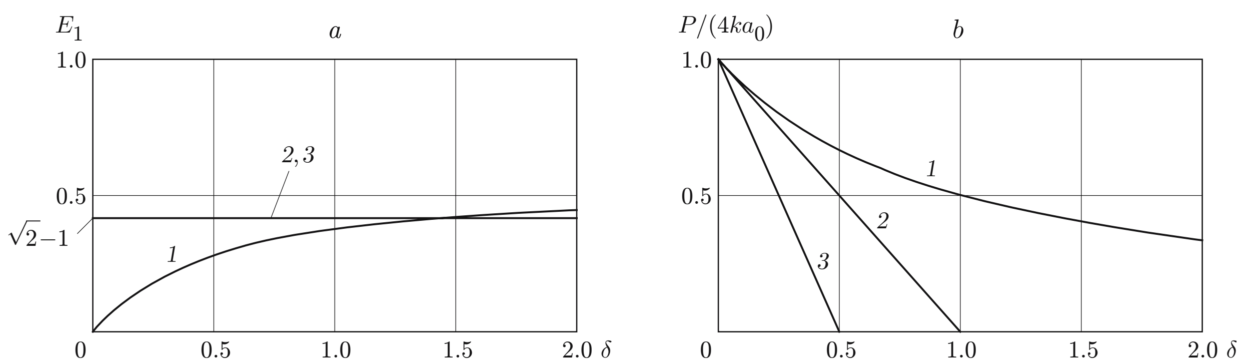

Fig. 2. The first principal value of the Almansi strain tensor E_1 (a) and the force $P/(4ka_0)$ necessary for sample deformation (b) versus dimensionless elongation: plastic flow with a uniform continuous field of displacement velocities (1); plastic flow with a discontinuous field of displacement velocities (Onat–Prager solution [9]) (2); plastic flow with a discontinuous field of displacement velocities with a double neck predicted by Onat and Prager (3).

The condition of choosing the direction of crack development with allowance for nonuniformity of the strain field in the vicinity of the crack tip can be formulated as follows: the plastic flow during the fracture is developed so that the increment of the work necessary for body deformation has the maximum value ($\delta A = \sup_{\varphi} \delta A$).

The process of crack propagation in plane and cylindrical samples was considered in detail in [6, 7, 10]. The corresponding solutions, which are extensions of the Onat–Prager solution, are plotted in Fig. 3. Such a character of the plastic flow yields an adequate description of the final stage of deformation of cylindrical samples [7] with formation of a neck, which is supported by experimental data.

Full Scheme of Deformation of a Plane Sample. At the first stage of sample extension, uniform deformation of the sample occurs at certain strains until a small-size macrocrack originates (Fig. 4a). After that, uniform deformation of the sample becomes impossible, and the plastic flow is described on the basis of an extension of the Onat–Prager solution (Fig. 4b). Macrocrack development up to sample separation into two fragments occurs. The basic sizes of the sample characterizing the process of its deformation are a_0, l_0, a_1, l_1, a_2, and l_2, which are the initial, intermediate, and final lengths and widths of the sample.

Determination of Fracture Constants. One of the basic experiments on determining the mechanical properties of materials is an experiment on extension of plane and cylindrical samples. The basic characteristic of fracture in this experiment is the dimensionless elongation $\delta = (l_2 - l_0)/l_0$ and the dimensionless constriction $\psi = (F_0 - F_2)/F_0$ of the sample during its fracture (F_0 and F_2 are the initial and final cross-sectional areas of the sample, respectively). For most materials, $\psi < 1$ and $F_2 > 0$ and the quantity F_2 can be considered as the area of the crack being formed.

We introduce two material constants: E_{**}, which is the value of the Almansi strain tensor E_1 corresponding to the end of the first (uniform) stage of sample deformation and determining the moment of origination of the macrocrack and the beginning of formation of a neck, and E_*, which is the value of E_1 at the tip of the macrocrack, determining the velocity of crack propagation.

The condition of incompressibility yields

$$l_1 a_1 = l_0 a_0. \tag{8}$$

At the first stage, the ends of the strip and the rigid regions MON and $M'ON'$ (see Fig. 3) move with identical absolute values of velocity V; hence, we have

$$a_1 - a_2 = l_1 - l_2. \tag{9}$$

From Eqs. (8) and (9), we can find l_1 and a_1:

$$l_1 = [A + B]/2, \qquad a_1 = [A - B]/2,$$

$$A = a_0(1 - \psi) + l_0(\delta + 1), \qquad B = \sqrt{a_0^2(1 - \psi)^2 + 2a_0 l_0(\delta - \psi - \psi\delta - 1) + l_0^2(\delta + 1)^2}.$$

The first principal value of the Almansi strain tensor E_{**} and the specific dissipation of energy corresponding to the end of the first stage of deformation are determined by expressions (7):

$$E_{**} = E_1 = (2\delta_1 + \delta_1^2)/(2(\delta_1 + 1)^2), \qquad W_{*1} = 2\ln(1 + \delta_1), \qquad \delta_1 = (l_1 - l_0)/l_0.$$

At the second stage of deformation (see Fig. 4b), the velocity of crack propagation dS/dt is determined by the velocity of propagation of the discontinuity line G. At a constant velocity of extension V, these quantities are related as

$$\frac{dS}{dt} = \sqrt{2}\, G, \qquad \frac{dS}{dt} = V\, \frac{a_2}{a_1 - a_2} = V\, \frac{1 - \psi_2}{\psi_2}, \qquad \psi_2 = \frac{a_1 - a_2}{a_1}.$$

Hence, we have

$$G = \frac{1}{\sqrt{2}}\, \frac{dS}{dt} = \frac{V(1 - \psi_2)}{\sqrt{2}\,(a_1/a_0 - (1 - \psi_2))}.$$

The volume density of energy dissipation at the line of discontinuity of displacement velocities is calculated by the formula $W_{*2} = 2\psi_2$ [10].

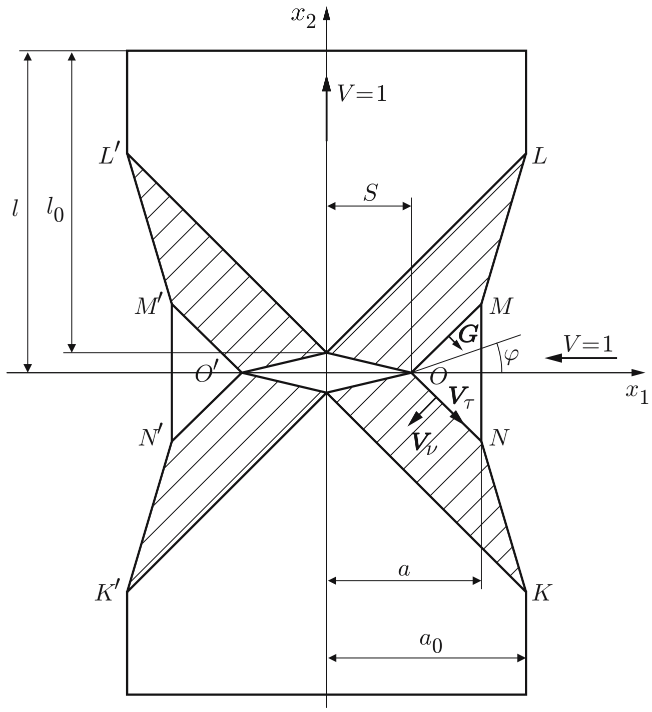

Fig. 3. Generalized Onat–Prager solution with a crack inside the sample.

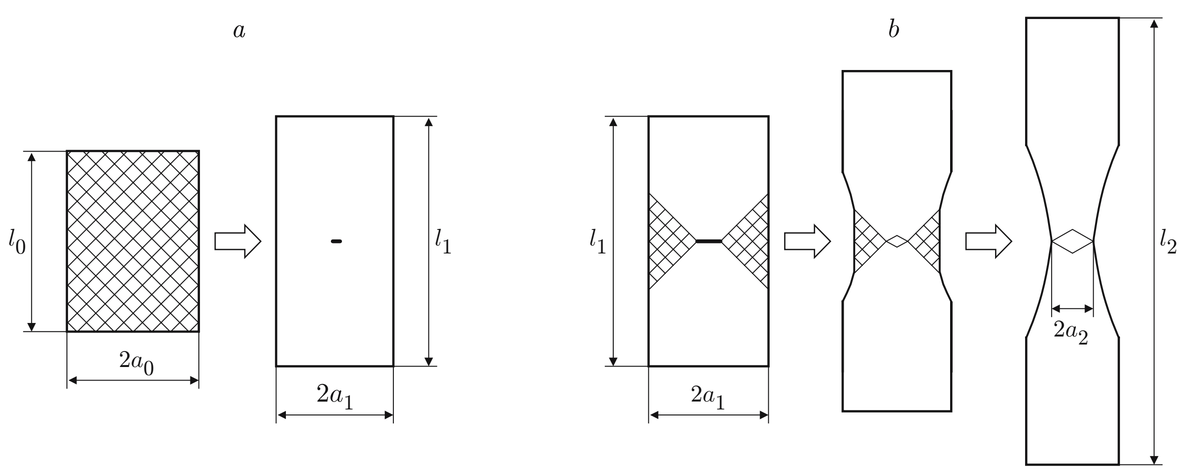

Fig. 4. Full scheme of deformation of a plane sample up to its fracture: in a uniform field of displacement velocities (a) and in a discontinuous field of displacement velocities with a propagating crack (b).

TABLE 1

Plastic Constants of Fracture

Material	δ, % [11]	ψ, % [11]	W_*	E_*	E_{**}
Aluminum alloys:					
AD0 (sheet)	35	80	1.6	0.384	0.168
AK8 (section)	7	15	0.3	0.129	0.046
AK8 (forging)	10	25	0.5	0.195	0.056
AD31 (section quenched and artificially aged)	17	70	1.4	0.364	0.035
AMG6 (plate, cold-worked, 18% in the longitudinal direction)	10	22	0.44	0.177	0.062
VD17 (strip pressed, quenched, and artificially aged, 60 mm)	10	19	0.38	0.157	0.068
AD33 (section pressed, quenched, and artificially aged)	12	30	0.6	0.223	0.066
Titanium allows:					
VT3-1 (forging)	14–20	45–60	0.9–1.2	0.291–0.340	0.056–0.085
VT6 (forging)	10–13	35–60	0.7–1.2	0.248–0.340	0.035–0.012
VT9 (forging)	8–14	25–45	0.5–0.9	0.195–0.291	0.035–0.056
VT14 (forging)	10–15	35–60	0.7–1.2	0.248–0.340	0.035–0.035

The components of the distortion tensor and the Almansi tensor are related by Eqs. (1) and the condition of incompressibility

$$A^2 + C^2 = 1 - 2E_{11}, \qquad B^2 + D^2 = 1 - 2E_{22},$$

$$AB + CD = -2E_{12}, \qquad AD - BC = 1,$$

where $A = \partial x_1^0/\partial x_1$, $B = \partial x_1^0/\partial x_2$, $C = \partial x_2^0/\partial x_1$, and $D = \partial x_2^0/\partial x_2$.

At the first stage, deformation is simple (the first principal directions of the Almansi and strain-rate tensors coincide with each other and with the x_2 axis); therefore, we have

$$B = 0, \quad C = 0, \quad E_{12} = 0, \quad E_{11} = E_1, \quad E_{22} = E_2,$$

$$A = \sqrt{1 - 2E_1}\,, \qquad D = \sqrt{1 - 2E_2}\,.$$

From here, according to Eq. (6), we obtain

$$\frac{\partial x_1^{0^+}}{\partial x_1} = A, \qquad \frac{\partial x_1^{0^+}}{\partial x_2} = W_2 D, \qquad \frac{\partial x_2^{0^+}}{\partial x_1} = 0, \qquad \frac{\partial x_2^{0^+}}{\partial x_2} = D.$$

The quantity E_* is determined by the expression

$$E_* = (1/4)(1 - A^2 - W_2^2 D^2 - D^2) + (1/2)\sqrt{(A^2 + W_2^2 D^2 - D^2)^2 + 4A^2 D^2}\,.$$

The total specific volume dissipation for particles deformed at the first and second stages is determined by the formula $W_* = W_{*1} + W_{*2}$.

The fracture constants for some structural materials are summarized in Table 1. The invariant tensor deformation and energy characteristics of fracture of structural materials allow correct application of experimentally determined quantities in calculating complicated structures and their elements by numerical methods.

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